Computing Fixed Points and Cycles for the Discrete Logistic Equation

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Abstract *The question of computing fixed points and cycles for the discrete logistic equation will be considered. A method will be presented for accomplishing these tasks. Illustrative cycles will be given to demonstrate use of the method. Along the way we will find out some interesting tidbits about the fixed points and cycles. We will also see the manner in which Maple can be used to explore a complex problem.*

1   Introduction

We are interested in the task of computing fixed points and N-cycles for the well known and much studied discrete logistic equation

$$x_{n+1} = kx_n \left(1 - x_n\right) \tag{1}$$

where k is a constant satisfying $0 \leq k \leq 4$. We are particularly interested in values of k between 3 and 4. Background and a detailed discussion of Eq. (1) may be found in [3]. A delightful elementary introduction may be found in [1]; and a more detailed mathematical treatment may be found in [2].

Finding N-cycles entails finding the fixed points of $F_N(x)$, the Nth composition of the function

$$F(x) = kx(1 - x) \tag{2}$$

with itself. We thus must find the solutions of

$$F_N(x) = x. \tag{3}$$

We must then determine the N-cycles determined by the fixed points. We are interested primarily in "true" (prime in the terminology of [3]) N-cycles, ones forming a sequence $x_0, \ldots, x_{N-1}$ where x_0 is a solution of Eq. (3), $F(x_{N-1}) = x_0$, and $N - 1$ is the first value for which the latter occurs. This is in general a formidable task since $F_N(x)$ is a polynomial of degree 2^N. While it is possible to use tools available in a *Computer Algebra System* (CAS) such as Maple 11 [4] to find the fixed points and corresponding cycles directly for relatively small values of N, say, $N < 5$, it is effectively not possible to do so for larger values of N. We will describe a method that we have used to help Maple accomplish these tasks for values of N up to 15 and for general values of k. From our discussion, it should become clear that constructing cycles without the use of a CAS such as Maple is simply intractable even for small values on N.

We note that `cobweb diagrams` [1] are used to display N-cycles (more generally, they are to depict the iterates for any sequence). These diagrams depict the relevant fixed points along with the graphs of $y = F(x)$ and the line $y = x$. Starting from a fixed point, locate the corresponding point on the curve $y = F(x)$. Then locate the corresponding point on the line $y = x$. Repeat this process to follow the formation of the N-cycle. This process is illustrated in the animated gif files attachments that are mentioned later in this paper.

It should be noted that our approach is somewhat empirical. However, the onset values of k at which cycles of length 1-8 first appear can be determined analytically. (See [6] and the references therein). In each such case, we were able to find cycles of the desired length for each of these onset values. We experimented to approximate the onset values for cycles of length 9-12. (Refer to the attached Maple worksheet for the approximate onset values we obtained. The worksheet also contains a summary of the numbers of cycles of length N that we were able to locate for different values of k.) To get an idea of where to start looking for cycles of these lengths, we used Sharkovskii's Theorem [2]-[3] (which involves ordering the natural numbers in a manner such that if any cycle of length N exists, all successors in the ordering also yield cycles of length N). Each of the cycles reported in this paper is consistent with Sharkovskii's ordering and with the onset values for shorter cycles. As a matter of interest, we further note that certain values of k yield cycles of several lengths. For example, $k = 3.9$ yields cycles of length 2-12 and $k = 3.75$ yields cycles of length 2-10. Finally, we note that it is instructive to use the attached worksheet to experiment with values on each side of the onset values to see how the cycles with longer lengths appear.

2   The Easiest Case, $k = 4$

Though the title of this section may seem surprising to readers familiar with the complexity of the logistic equation, the easiest case to handle occurs when $k = 4$. In this case, Eq. (3) has precisely 2^N real fixed points and N-cycles exist for each N. It is instructive to consider the manner in which the fixed points arise for this value of k.

Graphs of $F_1(x)$, $F_2(x)$, and $F_3(x)$ suggest there are precisely 2^N fixed points for each value of N and it suggests how we might go about finding the fixed points of $F_N(x)$. A simple calculation shows that the following hold

$$F_{N+1}(x) = 4F_N(x)\left(1 - F_N(x)\right) \tag{4}$$

$$F'_{N+1}(x) = 4F'_N(x)\left(1 - 2F_N(x)\right) \tag{5}$$

Eq. (4) and Eq. (5) may be used to show that the zeroes of $F_{N+1}(x) = 0$ consist of the zeroes of $F_N(x) = 0$ along with new values determined by $F_N(x) = 1$. The values for which $F_{N+1}(x) = 1$ are the values for which $F_N(x) = \frac{1}{2}$; and the zeroes of $F_{N+1}(x) = 0$ thus consist of 0 and 1 along with $2^N - 2$ zeroes with multiplicity 2. Each of the intervals with these distinct zeroes as endpoints contains exactly two fixed points of $F_{N+1}(x)$. As an example, the zeroes of $F_1(x)$ are 0 and 1. (Alternatively, the intervals between successive points with $F'_N(x) = 0$ each contain one fixed point.) These two values give rise to 0, $\frac{1}{2}$, $\frac{1}{2}$, and 1 as the zeroes of $F_2(x)$. These four values in turn yield the zeroes of $F_3(x)$ consisting of these four values along with the two multiplicity 2 values $\frac{1}{2} \overset{+}{-} \frac{\sqrt{2}}{4}$. The corresponding intervals each contain the two fixed points that can be located by subdividing the interval twice. Using the Maple `solve` command for $N = 2$ yields the fixed points $x = 0, x = \frac{3}{4}$, and $x = \frac{5}{8} \overset{+}{-} \frac{1}{8}\sqrt{5}$. For $N = 3$ it yields the fixed points (rounded for display purposes) 0, 0.11698, 0.18826, 0.41318, 0.61126, 0.75, 0.95048, and 0.96985. Figure 1 depicts a 2-cycle and a 3-cycle determined by these fixed points.

In principle, the Maple `solve` command can be used to solve the necessary polynomial equations in the above procedure. Unfortunately, for values of N as small as 4 or 5, use of the `solve` command is unsuccessful in some cases and in other cases an incorrect number of solutions is found. Use of the numeric `fsolve` command leads to similar results since `fsolve` recognizes a polynomial as a "known" function and employs symbolic procedures to solve polynomial equations. Difficulties using `solve` and `fsolve` led us to explore other ways to determine the fixed points. An alternate way to find the fixed points without directly solving polynomial equations is described in the next section.

3   Backward Compositions

If $x = F_n(x)$ then we may solve the equation $x = kx_{n-1}\left(1 - x_{n-1}\right)$ for the two values x_{n-1} for which $F\left(x_{n-1}\right) = x$. We refer to these two values as ancestors of x. (They are sometimes referred to as predecessors or preimages of x.) These values are given by

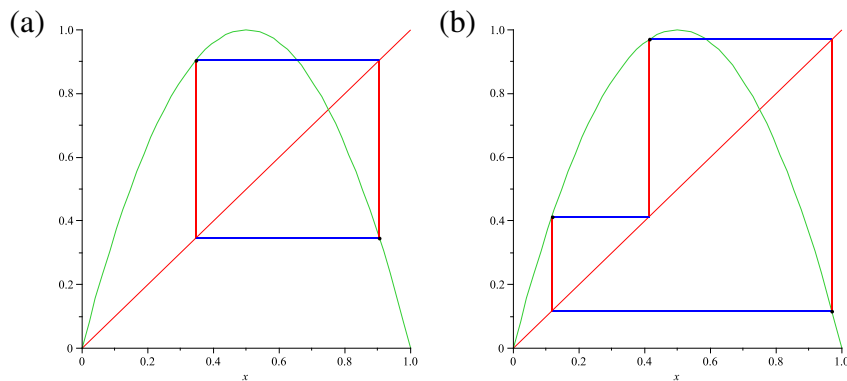$$x_{n-1} = \frac{1}{2}\left(1 \overset{+}{-} \sqrt{1 - \frac{4}{k}x}\right). \tag{6}$$

Figure 1: 2-cycle and 3-cycle for $k = 4$

Let us then define the functions

$$g_0(x) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{k}x}\right) \tag{7}$$

$$g_1(x) = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{k}x}\right) \tag{8}$$

For a given value of x, $g_0(x)$ and $g_1(x)$ yield the two immediate ancestors of x. The four ancestors of these values are given by the compositions $g_0 \circ g_0(x_{n-1})$, $g_0 \circ g_1(x_{n-1})$, $g_1 \circ g_0(x_{n-1})$, and $g_1 \circ g_1(x_{n-1})$. Continuing in this fashion we can locate the 2^N ancestors of x. We can then locate the fixed points of $x = F_N(x)$ by solving the equations

$$x = \Phi(x) \tag{9}$$

where $\Phi(x)$ is any of the 2^N nested square root compositions

$$g_{i_0} \circ \cdots \circ g_{i_N}(x)$$

and each i_j is 0 or 1 since any such solution satisfies Eq. (3). To locate the zeroes of the functions, we used a Maple adaptation `Zeromw` of the well-known `Zero` root finder [5] which uses a combination of the secant method and bisection. Although the fixed points of $F_N(x)$ become closely spaced for increasing N, they are well-separated by the above functions and are located easily. Figure 2 depicts a 6-cycle for $k = 4$. We obtained similar N-cycles for larger values of N. (Values of N in the range 12–15 were as large as our computer was willing to tolerate.) None are displayed here because higher order cycles look very much like the displayed 6-cycle due to the fact that some of the cycle points tend to cluster near 0 and 1 making it hard to distinguish between them in the plotted cycles.

Note that when $k = 4$ the domain of each of the backward compositions is the interval $[0, 1]$. This is as good as it gets. In general, the real domains of the functions $\Phi(x)$ are subsets of this interval, that is, some of the ancestors of x are complex. The key to making our approach work is to find these real domains so that we can locate the real ancestors of x. This will be discussed further in the next section.
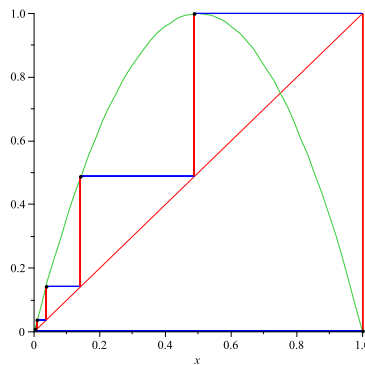
Figure 2: 6-cycle for $k = 4$

4 Backward Composition Domains

In order to find a real fixed point x using the procedure described above, x must belong to the domain of each of the nested square root functions used to build $\Phi(x)$. For $g_0(x)$ and $g_1(x)$ to be real, we must have $x \leq x_0 = \frac{k}{4}$. An interesting thing happens when we consider the cut points at which successive discriminants are equal to 0. Compositions of length two lead to the cut point

$$x_1 = x_0 \left(1 - (2x_0 - 1)^2\right). \tag{10}$$

Continuing with successively higher order compositions, we inductively find the sequence of cut points satisfying

$$x_j = x_{j-1} \left(1 - (2x_{j-1} - 1)^2\right), \ j = 1, \ldots, N - 1 \tag{11}$$

A simple calculation shows that

$$x_j = kx_{j-1} \left(1 - x_{j-1}\right), \tag{12}$$

That is, the cut points are the first N terms of a logistic sequence with initial iterate equal to $\frac{k}{4}$. This, perhaps surprising, tidbit is quite useful. The domains of each of the $\Phi(x)$ functions are (single) intervals of the form $[a, b]$ where a and b are two elements of this sequence with $a < b$ or of the form $[a, b]$ where $a = 0$ and b is an element of the cut sequence. For $k = 4$ note that the sequence is $1, 0, 0, \ldots$ leading to the interval $[0, 1]$ as the domain for each $\Phi(x)$. For other values of k, some of the cut points may be close together and in some cases the real domain of a particular composition degenerates to a single point. This is one reason it is necessary to locate the cut points since some of the fixed points tend to cluster in these small intervals.

Although it is relatively easy to obtain N-cycles for a good many values of N and k, our procedure has two drawbacks. It requires the solution of 2^N nonlinear equations making it impractical for large values of N due to the exponentially increasing complexity of the calculations. However, we have used it to locate cycles for many values of k and values of N up to 15 (although in some cases the calculation times for $N > 12$ are quite long). Figures 3–8 represent a miscellaneous sample of cycles of various lengths 2–13. A second nasty drawback can occur even for some small values of N. Unlike the case $k = 4$, it is necessary in some cases to subdivide the domains more than twice in order to find bracketing intervals containing the fixed points. In most cases 2–10 subdivisions are sufficient but in some situations many more subdivisions are required. The cases we have in mind are the ones near the "onset" values of k discussed in the next section.
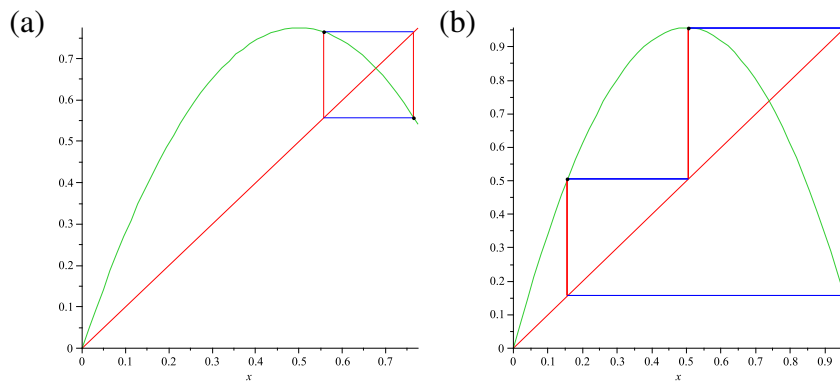
Figure 3: (a) 2-cycle for $k = 3.1$ (b) 3-cycle for $k = 3.83$
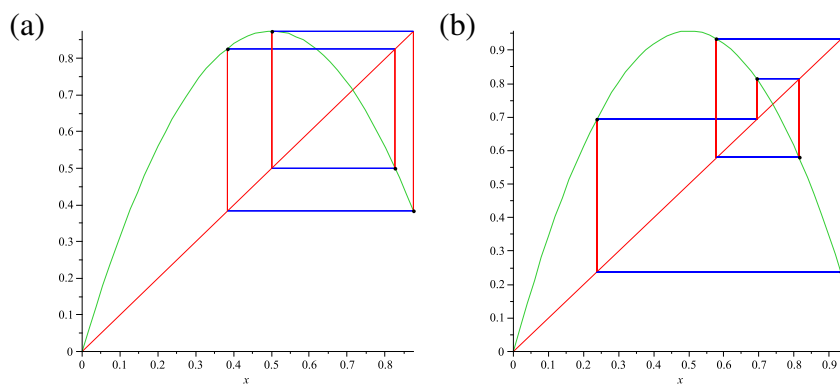

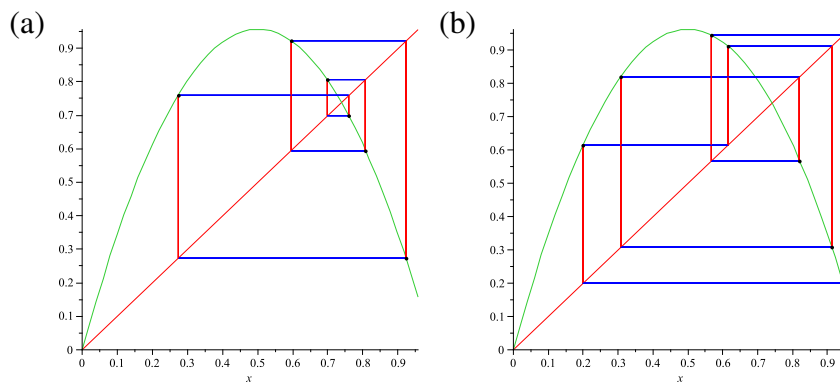Figure 4: (a) 4-cycle for $k = 3.5$ (b) 5-cycle for $k = 3.83$

Figure 5: (a) 6-cycle for $k = 1 + \sqrt{8}$ (b) 7-cycle for $k = 3.85$

5   Onset Values of k

Since Zeromw requires a sign change to locate zeroes, success of the procedure described above depends on being able to subdivide the domains of the composition functions to obtain bracketing
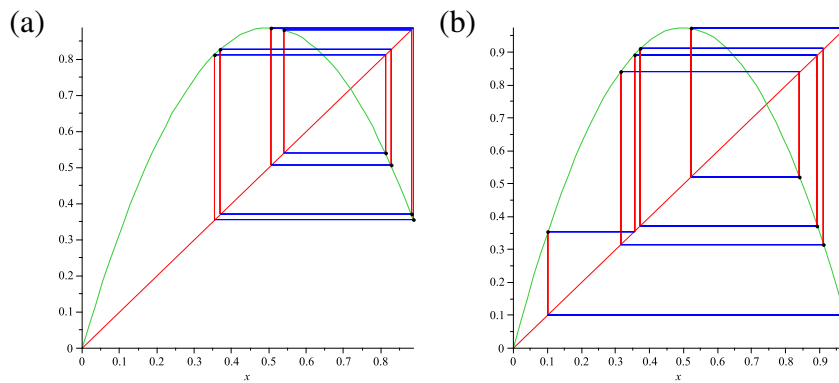
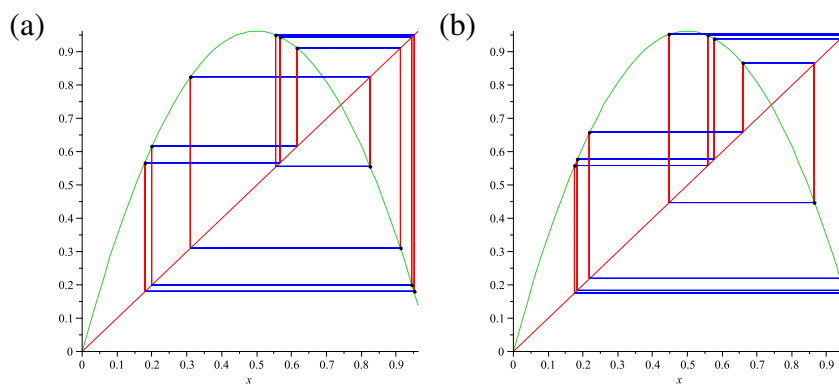Figure 6: (a) 8-cycle for $k = 3.55$ (b) 9-cycle for $k = 3.9$

Figure 7: (a) 10-cycle for $k = 3.85$ (b) 11-cycle for $k = 3.85$
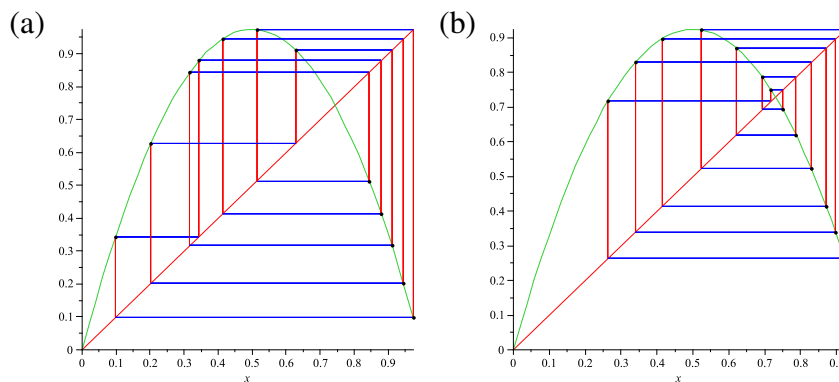

Figure 8: (a) 12-cycle for $k = 3.9$ (b) 13-cycle for $k = 3.7$

intervals for the fixed points. In some cases, our procedure is inefficient in locating zeroes with multiplicity greater than 1 or for nearby values of k for which there are very closely spaced zeroes near such a zero with multiplicity greater than 1. Such zeroes occur for special onset values of k at which cycles of longer lengths originate. Several such onset values are known (see [6] and the links

cited therein). Figures 9–12 depict cycles of length 3–8 corresponding to these onset values. Each cycle depicted was obtained by rounding the 54th digit of the corresponding exact value of k upward. This was done for convenience and since for some values of k, there is no true cycle of the desired length except for values of k strictly larger than the onset value.

We should note that handling the $k = 3$ onset value for 2-cycles in this manner was unsuccessful. Use of the method is prohibitive for smaller values of k near $k = 3$ due to the number of required subdivisions of the composition domains. However, in the Maple worksheet we used, it is possible to reduce the size of the composition domains if additional information is available. (Such information may be obtained by inspecting plots of the functions used in the root finding.) $k = 3.000000001$ was the smallest value of k for which we bothered to do this in order to compute a true 2-cycle. Since the `solve` command is able to find the fixed points of $F_2(x)$ for values of k near 3, it does a more satisfactory job finding the relevant fixed points than does the present method for values of k slightly larger than 3. We should also note that we were able to use the Maple `factor` to find the fixed points and onset cycles for these values of k (albeit at considerably more expense). The precise values of k used to produce the other onset cycles depicted in Figures 9–12 were:

```
3-cycle:
3.8284271247461900976033774484193961571393434375075389615
4-cycle:
3.4494897427831780981972840747058913919659474748065667013
5-cycle:
3.7381723752659623694302615596795319734441154404899193
6-cycle:
3.6265531616949737258772322520933174917094757857950247470
7-cycle:
3.701640764160349581824643789840889220144291589515206446
8-cycle:
3.5440903595519228536159659866048045405830998454444457368
```

Though somewhat more cumbersome (due to having to deal appropriately with poles), we found it possible to also produce these cycles in a different way. By replacing the equation $x - \Phi(x) = 0$ with

$$\frac{x - \Phi(x)}{1 - \Phi'(x)} = 0 \tag{13}$$

it was possible to locate higher multiplicity zeroes and very closely spaced zeroes using far fewer subdivisions of the composition domains. The cycles depicted in Figures 9–12 were obtained in this manner.

6 Suggested Student Explorations

The logistic equation is rife with interesting properties that beg to be investigated. Following are a few interesting questions that students might wish to explore.
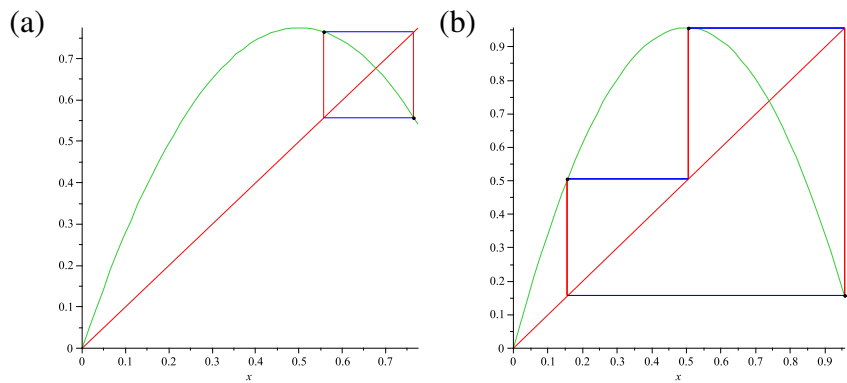
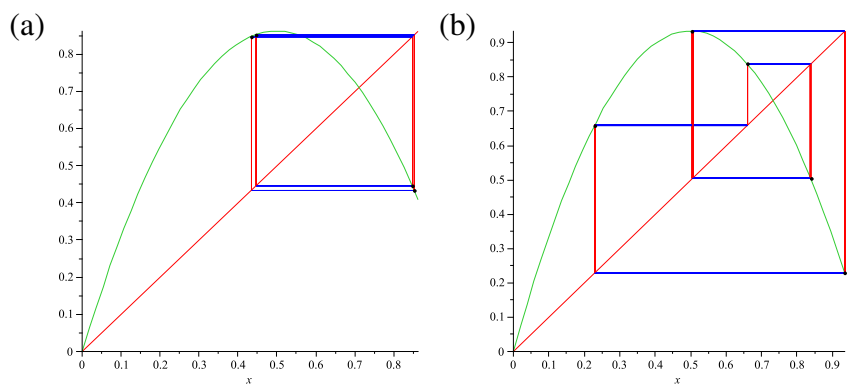Figure 9: (a) Onset 2-cycle for $k = 3$ (b) Onset 3-cycle for $k = 3.828$

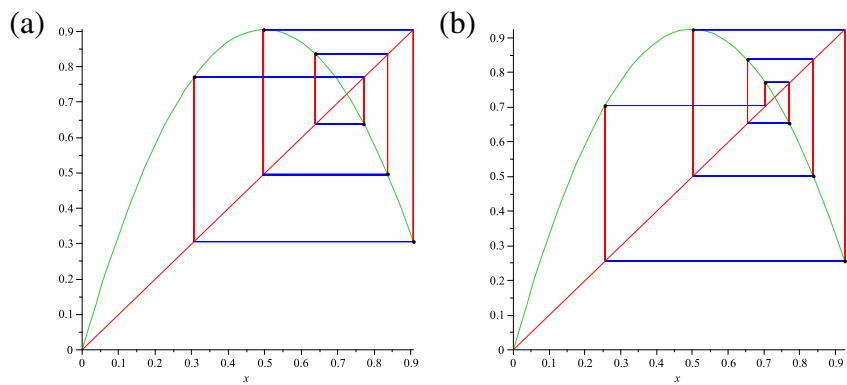Figure 10: (a) Onset 4-cycle for $k = 3.449$ (b) Onset 5-cycle for $k = 3.738$

Figure 11: (a) Onset 6-cycle for $k = 3.626$ (b) Onset 7-cycle for $k = 3.701$

6.1 Question 1

To develop an appreciation for the manner in which cycles of longer length develop near the onset values of k, you might wish to study the following cycles. Figure 13 depicts a 4-cycle for $k = 3.54$ and an 8-cycle for $k = 3.55$. Note how the additional close zeroes in the latter cycle get into the act
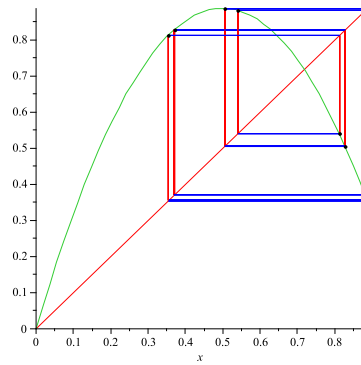
Figure 12: Onset 8-cycle for $k = 3.544$
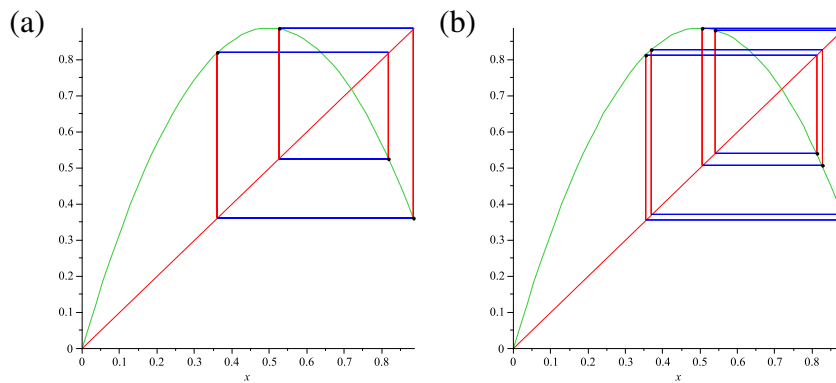
and prevent a 4-cycle from forming.

Figure 13: (a) $k = 3.54$ 4-cycle (b) $k = 3.55$ 8-cycle

6.2   Question 2

Invite students to experimentally find the "smallest" values of k that yield true 9-, 10-, 11-, and 12-cycles. They should discover that

- k = 3.68725 yields a 9-cycle.

- k = 3.60540 yields a 10-cycle.

- k = 3.68175 yields an 11-cycle.

- k = 3.58225 yields a 12-cycle.

The corresponding cycles are depicted in Figure 14. Animated versions of these cycles follow Figure 14. Depending on what is displayed, you need only play the cycle forward or backward using the control box.
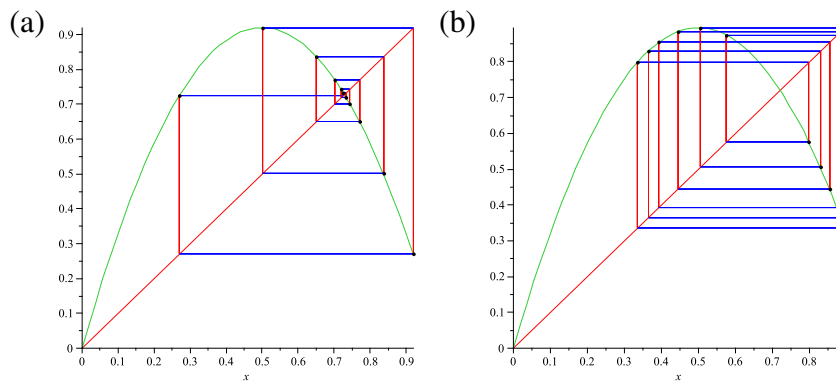
(a)

(b)

Figure 14: (a) 11-cycle and (b) 12-cycle

Animation of 10-cycle for $k = 3.60540$

6.3   Question 3

If you wish to really try your hand with Maple programming, there are several ways in which the attached worksheet can be improved. The manner in which the real domains of the backward composition functions are determined is relatively time consuming; consider using some care to improve it. Another hotspot is the processing of fixed points to determine which ones produce N-cycles. Perhaps the best place to improve performance is to consider using different numbers of subdivisions of the real domain intervals (that is, to use the `subdivs` array option in the program).

6.4   Question 4

Rather than solve the equations $x = \Phi(x)$, consider solving the equations $F(x) = \Phi(x)$ where $\Phi(x)$ is any of the 2^{N-1} nested square root compositions $g_{i_0} \circ \cdots \circ g_{i_{N-1}}(x)$. You will find that you can do

about as well with this approach as with the procedure used in this paper. Note: It is a simple matter to change the attached worksheet to accomplish this. Be sure to study the plots of the relevant root finding functions though.

6.5 Question 5

Investigate the question of solving the equations $x = \Phi(x)$ using fixed point iterations $x_{n+1} = \Phi(x_n)$ where $\Phi(x)$ is any of the 2^N nested square root composition $g_{i_0} \circ \cdots \circ g_{i_N}(x)$. To see why you don't pick up all of the fixed points, study the graphs of $\Phi'(x)$ and recall the basic condition for the convergence of a fixed point iteration.

6.6 Question 6

Explore the question of finding fixed points in the following manner. Using the functions $\Phi(x)$ it is easy to calculate all real ancestors for a given value of x. x is a fixed point if it is equal to one of its real ancestors. For a general value of x one can work with the function $f(x)$ defined to be the minimum distance between x and its real ancestors of x. A zero of this function yields the corresponding fixed point.

6.7 Question 7

Recall that the fixed points of $F_N(x)$ are the solutions of

$$F_N(x) = x. \tag{14}$$

$x = 0$ is always a solution. Other solutions satisfy

$$G_N(x) = \frac{F_N(x)}{x} = 1. \tag{15}$$

Note that the discontinuity at $x = 0$ can be removed using $G_N(0) = F_N'(0)$. Explore the question of finding the nonzero fixed points of $F_N(x)$ by finding solutions of this equation using Zeromw (or otherwise). Alternatively, find solutions of $F_N(x) = x$ without using this reduction. Refer to the discussion for $k = 4$ to determine possible bracketing intervals containing fixed points. What changes in that discussion are necessary for $k < 4$? What difficulties prevent this approach from being effective for general values of k?

7 Summary

In this paper we considered the question of computing fixed points and cycles for iterates of the logistic equation. A method was described for doing so which does not require directly solving high order polynomial equations. Cycles of various lengths were given to illustrate the use of the method. Finally, several possible student explorations were suggested.

8   Supplemental Electronic Material

A Maple worksheet Cycles.mw that can be used to obtain the results reported in this paper can be downloaded from the author's web site using the following link:

http://www.radford.edu/thompson/Logistic/index.html

References

[1] Blanchard, P., Devaney, R.L., and Hall, G.H., `Differential Equations, 3rd edition`, Thomson/Brooks–Cole, 2006.

[2] Devaney, R., `An Introduction to Chaotic Dynamical Systems, 2nd edition`, Addison–Wesley, 1989.

[3] Holmgren, R., *A First Course in Discrete Dynamical Systems*, Springer, 1996.

[4] Maple, Maplesoft, Waterloo Maple Inc., 615 Kumpf Avenue, Waterloo, Ontario, Canada, 2008.

[5] Shampine, L.F., Allen, R.C., and Preuss, S., *Fundamentals of Numerical Computing*, John Wiley & Sons, 1997.

[6] Weisstein, E.W, *Logistic Map*, MathWorld–A Wolfram Web Resource, `http://mathworld.wolfram.com/LogisticMap.html`.